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GENERALIZATION OF THE BIOT'S EQUATIONS FOR ACCOUNT OF FLUID SHEAR RELAXATION

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Based on the generalized variational principle [1-2] the system of the generalized Biot's equations is derived for consistent account of fluid shear relaxation. Account of shear viscosity relaxation leads to existence of a couple shear propagation modes additionally to a couple of longitudinal modes as in the Biot's approach. At this the one shear mode is an acoustical one, while the other shear mode has diffusive behavior at low frequencies. Phase velocity and attenuation factor for the second shear mode linearly depend on frequency in the low frequency limit that is different from analogous behavior of diffusive longitudinal mode with the root frequency behavior of analogous values.

Heavy-oil is currently one of the top themes in the industry worldwide. Properties of heavy-oil reservoirs show rheological behavior and may vary dramatically from area to area, and they correct description in porous medium is important for optimization of recovery methods. Account of viscosity relaxation is usually introduced through generalization of the Biot's operator for dissipation function. But this approach is not completely consistent because it is not based on variational principle being the initial basis of Biot's theory [3].

Generalization of the Biot's equations with account of relaxation behavior of fluid, filling a porous space, can be given on the basis the generalized variational principle for dissipative hydrodynamics [1-2]. It joints the Hamilton's and Osager's variational principles for mechanical and dissipative hydrodynamical systems in terms of the mechanical and thermal displacement fields. A system of hydrodynamical equations for these fields is derived from the stationary condition for an action with a Lagrangian in the form of the difference between the kinetic and free energies which include the time integral of the dissipation function containing quadratic forms of all the terms.

The kinetic energy has the same form as in the Biot's theory

$$2K(\dot{\vec{u}}) = \rho_{11}\dot{\vec{u}}_1^2 + 2\rho_{12}\dot{\vec{u}}_1\dot{\vec{u}}_2 + \rho_{22}\dot{\vec{u}}_2^2$$

In expression for the free energy the additional terms are appeared which are dealt with an internal parameter in accordance to Mandelshtam-Leontowich approach. As it is possible to show, for account of shear viscosity the fluid should be considered initially as an elastic medium. With these remarks we can write the free energy of a porous medium in the following form

$$\begin{aligned} 2F = {} & 2\mu_{11}\varepsilon 1_{ik}^2 + \lambda_{11}\varepsilon 1_{ll}^2 + 2\mu_{22}\varepsilon 2_{ik}^2 + \lambda_{22}\varepsilon 2_{ll}^2 + \\ & + 2\mu_{12}\varepsilon 1_{ik}\,\varepsilon 2_{ik} + 2\lambda_{12}\varepsilon 1_{ll}\,\varepsilon 2_{kk} + a_1\xi_{ll}^2 + a_2\xi_{ik}^2 + \\ & + 2b_1\xi_{kk}\varepsilon 2_{ll} + 2b_2\xi_{ik}\varepsilon 2_{ki} + 2c_1\xi_{ik}\varepsilon 1_{ll} + 2c_2\xi_{ik}\varepsilon 1_{ki} \end{aligned} \tag{1}$$

Here the deformation tensors of elastic matrix and fluid are denoted by $\varepsilon 1_{ik}$ and $\varepsilon 2_{ik}$, and ξ_{ik} is a tensor internal parameter.

The dissipation function includes as the terms from dissipation inside the viscous fluid as from the relative phases motion. At this it has to vanish in the equilibrium state, when the internal macroscopic motion and relaxation processes are absent.

$$2D(\dot{\vec{u}}_1, \dot{\vec{u}}_2, \dot{\xi}_{ik}) = \beta(\dot{\vec{u}}_1 - \dot{\vec{u}}_2)^2 + \gamma_1\dot{\xi}_{ll}^2 + \gamma_2\dot{\xi}_{ik}^2 . \tag{2}$$

It is easy to see, that at zero values of coefficients responsible for inter-phase interaction λ_{12}, μ_{12}, β, ρ_{12}, the represented potentials describe separately an elastic medium and a viscous fluid with relaxation.

We are interesting to understand the role of shear relaxation at inter-phase interaction. By this reason and to avoid cumbersome formulae we will keep, farther, in (1) the terms including parameter ξ_{ik}, but not its convolution ξ_{ll}, considering the case $a_1 = 0$, $b_1 = 0$, $c_1 = 0$, $\gamma_1 = 0$.

The motion equations for this system will have the forms

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{\vec{u}}_1} - \nabla \frac{\partial F}{\partial \nabla \vec{u}_1} = -\frac{\partial D}{\partial \dot{\vec{u}}_1} \qquad \frac{d}{dt}\frac{\partial K}{\partial \dot{\vec{u}}_2} - \nabla \frac{\partial F}{\partial \nabla \vec{u}_2} = -\frac{\partial D}{\partial \dot{\vec{u}}_2} \qquad \frac{\partial D}{\partial \dot{\xi}_{ik}} - \nabla \frac{\partial F}{\partial \xi_{ik}} = 0$$

After substitution of the potentials (1), (2) in these equations, they reduce to the forms

$$\rho_{11}\ddot{\vec{u}}_1 + \rho_{12}\ddot{\vec{u}}_2 - \mu_{11}\Delta\vec{u}_1 - \mu_{12}\Delta\vec{u}_2 - (\lambda_{11} + \mu_{11})\nabla(\nabla\vec{u}_1) - \\ -(\lambda_{12} + \mu_{12})\nabla(\nabla\vec{u}_2) = -\beta(\dot{\vec{u}}_1 - \dot{\vec{u}}_2) + c_2\frac{\partial \xi_{ik}}{\partial x_k}, \tag{3}$$

$$\rho_{22}\ddot{\vec{u}}_2 + \rho_{12}\ddot{\vec{u}}_1 - \mu_{22}\Delta\vec{u}_2 - \mu_{12}\Delta\vec{u}_1 - (\lambda_{22} + \mu_{22})\nabla(\nabla\vec{u}_2) - \\ -(\lambda_{12} + \mu_{12})\nabla(\nabla\vec{u}_1) = -\beta(\dot{\vec{u}}_2 - \dot{\vec{u}}_1) + b_2\frac{\partial \xi_{ik}}{\partial x_k}, \tag{4}$$

$$\gamma_2\dot{\xi}_{ik} + a_2\xi_{ik} + b_2\varepsilon 2_{ik} + c_2\varepsilon 1_{ik} = 0, \tag{5}$$

As it was shown, at absence of the elastic – relaxation connections in the fluid: $b_2 = 0$, $c_2 = 0$, the first two equations (3), (4) are the analogy of the well known Biot's equations for two elastic continuums. If additionally to put $\mu_{22} = 0$ and $\mu_{12} = 0$, we obtain exactly the Biot's equation system. It allows us to determine all coefficients in quadratic forms by direct comparison of these equations.

The solution of the kinetic equation (5) is represented as:

$$\xi_{ik} = -\frac{1}{\gamma_2}\int_{-\infty}^{t} e^{-\frac{a_2}{\gamma_2}(t-t')}\left(b_2\varepsilon 2_{ik}(t') + c_2\varepsilon 1_{ik}(t')\right)dt'$$

After substitution of this solution in the last items in (3) and (4), the lasts can be written in the forms

$$\rho_{11}\ddot{\vec{u}}_1 + \rho_{12}\ddot{\vec{u}}_2 - \mu_{11}\Delta\vec{u}_1 - \mu_{12}\Delta\vec{u}_2 - (\lambda_{11} + \mu_{11})\nabla(\nabla\vec{u}_1) - (\lambda_{12} + \mu_{12})\nabla(\nabla\vec{u}_2) = -\beta(\dot{\vec{u}}_1 - \dot{\vec{u}}_2) + \\ + \frac{1}{2}\frac{c_2}{\gamma_2}\int_{-\infty}^{t} dt' e^{-\frac{a_2}{\gamma_2}(t-t')}\left(b_2\left(\Delta\vec{u}_2 + \nabla(\nabla\vec{u}_2)\right) + c_2\left(\Delta\vec{u}_1 + \nabla(\nabla\vec{u}_1)\right)\right) \tag{3b}$$

$$\rho_{22}\ddot{\vec{u}}_2 + \rho_{12}\ddot{\vec{u}}_1 - \mu_{22}\Delta\vec{u}_2 - \mu_{12}\Delta\vec{u}_1 - (\lambda_{22} + \mu_{22})\nabla(\nabla\vec{u}_2) - -(\lambda_{12} + \mu_{12})\nabla(\nabla\vec{u}_1) = -\beta(\dot{\vec{u}}_2 - \dot{\vec{u}}_1) + \\ + \frac{1}{2}\frac{b_2}{\gamma_2}\int_{-\infty}^{t} dt' e^{-\frac{a_2}{\gamma_2}(t-t')}\left(b_2\left(\Delta\vec{u}_2 + \nabla(\nabla\vec{u}_2)\right) + c_2\left(\Delta\vec{u}_1 + \nabla(\nabla\vec{u}_1)\right)\right)\text{(4b)} \tag{4b}$$

Thus, the equation system (3b), (4b) is the generalization of the Biot's equations system for the case, when the porous fluid is a relaxing medium.

Let's consider now the consequences following from this generalization. By use of expansion of vector displacement fields $\vec{u}_1$ and $\vec{u}_2$ onto longitudinal and transversal parts

$$\vec{u}_1 = grad\varphi_1 + rot\vec{\psi}_1 \qquad \vec{u}_2 = grad\varphi_2 + rot\vec{\psi}_2$$

and by consequent application of operators *div* and *rot* to the equations (3b), (4b), it is possible to show that the scalar and vector potentials of the displacements satisfy to the following equation system (for the Fourier's representation by time variable ($t \to \omega$))

$$\hat{A}\Delta\vec{\varphi} + \hat{B}\vec{\varphi} = 0 \qquad \hat{C}rotrot\vec{\psi} - \hat{B}\vec{\psi} = 0 \tag{5}$$

where for compactness the following vectors are introduced in the field spaces

$$\vec{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \qquad \vec{\psi} = \begin{pmatrix} \vec{\psi}_1 \\ \vec{\psi}_2 \end{pmatrix},$$

and the matrix $\hat{A}$, $\hat{B}$ and $\hat{C}$ have the following representations

$$\hat{A} = \begin{pmatrix} \tilde{\lambda}_{11} & \tilde{\lambda}_{12} \\ \tilde{\lambda}_{12} & \tilde{\lambda}_{22} \end{pmatrix} \qquad \hat{C} = \begin{pmatrix} \tilde{\mu}_{11} & \tilde{\mu}_{12} \\ \tilde{\mu}_{12} & \tilde{\mu}_{22} \end{pmatrix} \qquad \hat{B} = \begin{pmatrix} \omega^2\rho_{11} + i\omega\beta & \omega^2\rho_{12} - i\omega\beta \\ \omega^2\rho_{12} - i\omega\beta & \omega^2\rho_{22} + i\omega\beta \end{pmatrix}$$

and for shortness the following notation are introduced

$$\tilde{\lambda}_{jk} = \lambda_{jk} + 2\mu_{jk} - \frac{b_2^{j+k-2} c_2^{4-j-k}}{i\omega\gamma_2 + a_2}, \qquad \tilde{\mu}_{jk} = \mu_{jk} - \frac{1}{2}\frac{b_2^{j+k-2} c_2^{4-j-k}}{i\omega\gamma_2 + a_2} \tag{6}$$

As it is possible to see, the complete equation system (5) is split on two independent subsystems for longitudinal and transversal potentials, which describe, correspondingly, propagation of longitudinal and transversal waves.

Even from the structure of the equations (5) it is seen that at account of visco-elasticity of a fluid it is possible the existence of two independent longitudinal modes, as for the Biot's approach, as well as two independent shear modes, that is the generalization of the Biot's approach.

Indeed, each of the matrix equations (5) can be diagonalized by linear transformation of variables. For example by introducing of linear combination $\vartheta = \varphi_1 + \varepsilon\varphi_2$, the first of the equations (5) can be split on a couple of independent Helmholtz's equations

$$\Delta\vartheta_{1,2} + Kl_{1,2}^2(\omega)\vartheta_{1,2} = 0 \tag{7}$$

where the squares of the wave numbers are represented in the low frequency approximation by expressions

$$Kl_1^2(\omega) = \frac{\omega^2(\rho_{11} + \rho_{12})}{\tilde{\lambda}_{11} + \tilde{\lambda}_{12}} \quad \text{and} \quad Kl_2^2(\omega) = i\omega\beta\frac{\tilde{\lambda}_{11} + 2\tilde{\lambda}_{12} + \tilde{\lambda}_{22}}{\tilde{\lambda}_{11}\tilde{\lambda}_{22} - \tilde{\lambda}_{12}^2} \tag{8}$$

From these expressions it is seen that the first mode with the wave number (8) is an acoustical one, while the second mode is the diffusive one. The same behavior of longitudinal modes follows from the Biot's approach [3].

Let's consider the modes appeared after digitalization of the second equation (5) for the shear waves. Again by introduction of the linear combination of fields $\vec{\varsigma} = \vec{\psi}_1 + \tau\vec{\psi}_2$ this equation can be split onto two independent Helmholtz's equations

$$rotrot\vec{\varsigma}_{1,2} - Ks_{1,2}^2(\omega)\vec{\varsigma}_{1,2} = 0 \tag{9}$$

where squares of wave numbers are represented similar to (8)

$$Ks_1^2(\omega) = \frac{\omega^2(\rho_{11}+\rho_{12})}{\tilde{\mu}_{11}+\tilde{\mu}_{12}} \quad \text{and} \quad Ks_2^2(\omega) = i\omega\beta\frac{\tilde{\mu}_{11}+2\tilde{\mu}_{12}+\tilde{\mu}_{22}}{\tilde{\mu}_{11}\tilde{\mu}_{22}-\tilde{\mu}_{12}^2} \tag{10}$$

By the form the expressions for wave numbers of shear modes (10) look like as expressions for wave numbers of longitudinal modes (8) and in general case it is possible to wait the similar properties of shear modes as for the longitudinal ones. It means that at low frequencies the first mode (10) has to be acoustical one, while the second mode in (10) has to be diffusive one.

However, it is needed to remember that we described a viscous fluid as an elastic medium with relaxation. To obtain the viscous fluid from an elastic medium we have to put

$$\mu_{22} = \frac{b_2^2}{2a_2} \quad \text{and} \quad \mu_{12} = \frac{b_2 c_2}{2a_2} \tag{11}$$

At this the acoustical mode (10) continues to be acoustical one, while the behavior of the diffusive mode changes. Indeed in the low frequency limit at the conditions (11) we will have

$$Ks_1^2(\omega) = \frac{\omega^2(\rho_{11}+\rho_{12})}{\mu_{11}-c_2^2/2a_2^2} \qquad Ks_2^2(\omega) = \frac{\beta}{\gamma_2}\left(\frac{a_2}{b_2}\right)^2 + i\omega\delta \tag{12}$$

It is follows from (12) that the second mode occurs the diffusive one, but with more strong (lineal by frequency) dispersion of phase velocity

$$c_{s2}(\omega) = \omega\sqrt{\frac{\gamma_2}{\beta}\frac{b_2}{a_2}},$$

which value depends on ratio of dissipative factors γ_2 и β . Attenuation coefficient also occurs to be lineal by frequency

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