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GERTLER INSTABILITY OF COMPRESSIBLE NON-EQUILIBRIUM BOUNDARY LAYER

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The stability of spatial disturbances of compressible boundary layer of non-equilibrium gas on slow-bowed surface was investigated. The equations for description of Gertler instability in non-equilibrium media has been founded. Increment of this instability in non-equilibrium boundary layer was defined in dependence of degree of non-equilibrium and Gertler parameter. In was shown that its value is sufficiently raised with growth of degree of non-equilibrium.

The sufficient influence on flows near concave surfaces was exerted by centrifugal force. In result the boundary layer may becomes unstable in relation to longitudinal vortexes [1].

The particular feature of such instability is small intensity of Gertler vortexes and slow downstream amplification. Since we must taking into account the nonconcurrency effects initially. In equilibrium media the Gertler instability was investigated in [2]. In non-equilibrium media the corresponding investigations are absent.

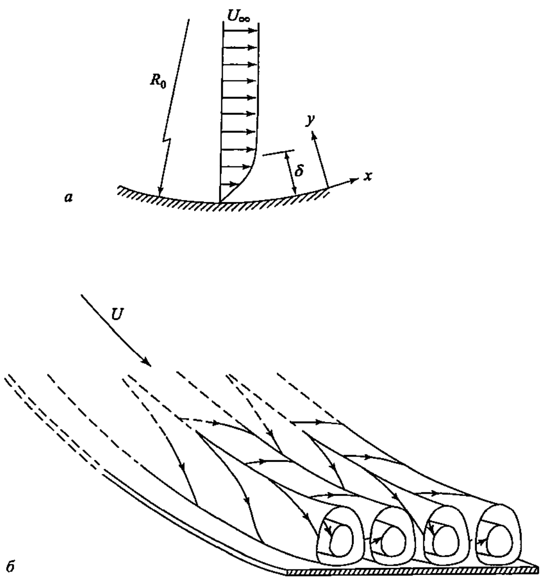

Fig. 1. The scheme of base flow

Let us consider as main flow the two-dimensional boundary layer on concave surface, which charachterized by two main small parameters: viscous parameter $\varepsilon_0 = R_0^{-1}$, where R_0 is radius of curvature and parameter of curvature $k = (K\delta)^{1/2}$, where K is curvature of surface. Let us supposed the limit of infinitesimal values ε and k and finite Gertler parameter $k/\varepsilon_0 = G$. In this case we can used as initial approach the approach of plane boundary layer. The scheme of flow is shown on fig. 1.

The set of input equations of relaxation gas dynamics, which includes the equations of continuity, Navier--Stokes, of state for gas, of heat transfer, and the relaxation equation. Linearisation of initial system we leaded in form $u = U(x,y) + u_1$, $v = V(x,y) + v_1$, $w = w_1$, $T = T_0(x,y) + T$, $\rho = \rho_0(x,y) + \rho_1$, $P = P_0 + P_1$, $E = E_0(x,y) + E_1$. Let us suppose [3], that disturbances have a following orders: $u_1 = O(1), v_1 = O(\varepsilon), w_1 = O(\varepsilon), P_1 = O(\varepsilon^2), T_1 = O(1)$, where $u_1 = \hat{u}_1(y)\exp\left(\int \sigma dx\right)\cos\beta z$ etc. In main order of smallness we have

$$\left[\rho_0\left(\sigma U + \frac{\partial U}{\partial x}\right) + \mu\beta^2\right]\hat u_1 + \left[\rho_0 V - \frac{\partial\mu}{\partial y}\right]\frac{d\hat u_1}{dy} - \mu\frac{d^2\hat u_1}{dy^2} - \rho_0\frac{\partial U}{\partial y}\hat v_1 + \left(U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y}\right)\hat\rho_1 -$$

$$- \frac{\partial}{\partial y}\left(\frac{\partial\mu}{\partial T}\frac{\partial U}{\partial y}\right)\hat T_1 - \frac{\partial\mu}{\partial T}\frac{\partial U}{\partial y}\frac{d\hat T_1}{dy} = 0,$$

$$\left[\rho_0\left(\frac{\partial V}{\partial x}\right) + 2UG^2 + \frac{2\sigma}{3}\frac{\partial\mu}{\partial y}\right]\hat u_1 + \left[\frac{\sigma\mu}{3} - \frac{\partial\mu}{\partial x}\right]\frac{d\hat u_1}{dy} + \left[\rho_0\left(\sigma U + \frac{\partial V}{\partial y}\right) + \mu\beta^2\right]\hat v_1 + \left[\rho_0 V - \frac{4}{3}\frac{\partial\mu}{\partial y}\right]\frac{d\hat v_1}{dy} -$$

$$-\frac{4\mu}{3}\frac{d^2\hat v_1}{dy^2} + \frac{2}{3}\frac{\partial\mu}{\partial y}\beta\hat w_1 - \mu\beta\frac{d\hat w_1}{dy} + \frac{d\hat P_1}{dy} + \left(U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y}\right)\hat\rho_1 + \left[\frac{2}{3}\frac{\partial\mu}{\partial T}\frac{\partial U}{\partial x} - \frac{4}{3}\frac{\partial\mu}{\partial T}\frac{\partial V}{\partial y}\right]\frac{d\hat T_1}{dy} -$$

$$-\left[\frac{U^2G^2\rho_0}{T_0} + \frac{1}{3}\frac{\partial\mu}{\partial T}\frac{\partial^2 U}{\partial x\partial y} + \frac{2}{3}\frac{\partial^2\mu}{\partial T^2}\frac{\partial T}{\partial y}\frac{\partial U}{\partial x} + \frac{4}{3}\frac{\partial}{\partial y}\left(\frac{\partial\mu}{\partial T}\frac{\partial V}{\partial y}\right) + \sigma\frac{\partial\mu}{\partial T}\frac{\partial U}{\partial y}\right]\hat T_1 = 0,$$

$$\beta\left[\frac{\sigma\mu}{3} + \frac{\partial\mu}{\partial T}\frac{\partial T}{\partial x}\right]\hat u_1 + \beta\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial y}\hat v_1 + \frac{\mu\beta}{3}\frac{d\hat v_1}{dy} + \left[\rho_0\sigma U + \frac{4\mu\beta^2}{3}\right]\hat w_1 + \left[\rho_0 V - \frac{\partial\mu}{\partial T}\frac{\partial T}{\partial y}\right]\frac{d\hat w_1}{dy} - \mu\frac{d^2\hat w_1}{dy^2} -$$

$$-\beta\hat P_1 - \frac{2\beta}{3}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\hat T_1 = 0, \qquad (1)$$

$$\left(\sigma\rho_0 + \frac{\partial\rho_0}{\partial x}\right)\hat u_1 + \rho_0\frac{d\hat v_1}{dy} + \frac{\partial\rho_0}{\partial y}\hat v_1 + \rho_0\beta\hat w_1 + \left(\sigma U + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right)\hat\rho_1 + V\frac{d\hat\rho_1}{dy} = 0,$$

$$(\sigma\rho_0 +)\frac{\partial T_0}{\partial x}\hat u_1 - \frac{2\mu}{C_{V\infty}}\frac{\partial U}{\partial y}\frac{d\hat u_1}{dy} + \frac{\partial T_0}{\partial y}\hat v_1 + \left[\sigma U + \frac{1}{C_{V\infty}}\left(\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right) - \frac{\partial\mu}{\partial T}\left(\frac{\partial U}{\partial y}\right)^2\right) - \frac{1}{Pr}\frac{\partial}{\partial y}\left(\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial y}\right) + \frac{\mu\beta^2}{Pr}\right]\hat T_1 +$$

$$+\left[\rho_0 V - \frac{2}{Pr}\frac{\partial\mu}{\partial T}\frac{\partial T}{\partial y}\right]\frac{d\hat T_1}{dy} - \frac{\mu}{Pr}\frac{d^2\hat T_1}{dy^2} + \frac{C_K\hat T_1 - \hat E_1}{C_{V\infty}\tau} + \frac{q}{C_{V\infty}}\left(\tau_\rho\frac{\hat\rho_1}{\rho_0} + \tau_T\frac{\hat T_1}{T_0}\right) = 0,$$

$$\frac{\partial E_0}{\partial x}\hat u_1 + \frac{\partial E_0}{\partial y}\hat v_1 + \sigma U\hat E_1 + V\frac{d\hat E_1}{dy} - \frac{C_K\hat T_1 - \hat E_1}{\tau} - q\left(\tau_\rho\frac{\hat\rho_1}{\rho_0} + \tau_T\frac{\hat T_1}{T_0}\right) = 0,$$

where $x = x/\delta$, $y = y/\delta$, $z = z/\delta$, $t = t\delta/U_\infty$, wave number $k = k\delta$, frequency $\alpha c = \omega U_\infty/\delta$. Here $\eta = \eta/\eta_\infty$, $Pr = \eta_\infty/\kappa_\infty$, $Re = U_\infty\delta/\eta_\infty$, $M_\infty = U_\infty/u_{S\infty}$ are Prandtl, Reynolds and Mach numbers of undisturbed flow, E is vibrational energy, $E_e = E\big|_{T_V = T_0}$ ins equilibrium value τ is time of VT – relaxation, Q Q is the power of external pumping source per molecule, $\mu = \eta/Re$.

System (1) must be supplied by boundary conditions

$$\hat u_1(0) = \hat v_1(0) = \hat w_1(0) = 0,\ \hat u_1(\infty) = \hat v_1(\infty) = \hat w_1(\infty) = \hat P_1(\infty) = \hat T_1(\infty) = 0. \qquad (2)$$

Let us consider limit $\beta \to \infty$, where values $G = O(\beta^4)$, $\sigma = O(\beta^2)$. The analysis of system (1) shows that $\hat v_1 = O(\beta^2\hat u_1)$, $\hat T_1 = O(\hat u_1)$, [3]. Since in main order of smallness we have the system of algebraic equations

$$\left(\frac{\sigma U}{T_0} + \mu\right)\hat u_1 + \frac{U'}{T_0}\hat v_1 = 0,\ \frac{2UG^2}{T_0}\hat u_1 + \left(\frac{\sigma U}{T_0} + \mu\right)\hat v_1 - \frac{U^2G^2}{T_0}\hat T_1 = 0,$$

$$\frac{T_0'}{T_0}\hat v_1 + \left(\frac{\sigma U}{T_0} + \frac{\mu}{Pr}\right)\hat T_1 + \frac{C_K\hat T_1 - \hat E_1}{\tau T_0 C_{V\infty}} + q\left(\tau_T - \tau_\rho\right)\frac{\hat T_1}{C_{V\infty}T_0^2} = 0,\quad \frac{E_0'}{T_0}\hat v_1 + \frac{\sigma U}{T_0}\hat E_1 - \frac{C_K\hat T_1 - \hat E_1}{\tau T_0 C_{V\infty}} - q\left(\tau_T - \tau_\rho\right)\frac{\hat T_1}{C_{V\infty}T_0^2} = 0, \quad (3)$$

$$\frac{dv_1^{(0)}}{d\xi} + w_1^{(0)} = 0,\ \mu\frac{dv_1^{(0)}}{d\xi} + \left(\frac{4\mu}{3} + \frac{\sigma U}{T_0}\right)w_1^{(0)} - P_1^{(0)} = 0.$$

After [3] we supposed that $\widehat{u}_1(\xi) = u_1^{(0)}(\xi) + \theta^{1/2} u_1^{(1)}(\xi) + \theta u_1^{(2)}(\xi) + \ldots$, etc. $G^2 = \dfrac{g_0 + \theta^{1/2} g_1 + \theta g_2 + \ldots}{\varepsilon^4}$, где $\xi = \dfrac{y - y_+}{\varepsilon^{1/2}}$, $\varepsilon = \beta^{-1}$ too. After substitution of these expansions to initial system (1) in main order we have

$$q_1 u_1^{(0)} + \frac{U'}{T_0} v_1^{(0)} = 0, \quad \frac{2UG^2}{T_0} u_1^{(0)} + q_1 v_1 - \frac{U^2 G^2}{T_0} T_1^{(0)} = 0, \tag{4}$$

$$\left[(1 + \sigma_0 U\tau) \frac{T_0'}{T_0} + \frac{E_0'}{C_{V\infty} T_0} \right] v_1^{(0)} + \left[(1 + \sigma_0 U\tau) q_2 + \frac{\sigma_0 U}{T_0} \frac{C_{P0} - C_{P\infty}}{C_{V\infty}} \right] T_1^{(0)} = 0,$$

where $q_1 = \dfrac{\sigma U}{T_0} + \mu$, $q_2 = \dfrac{\sigma U}{T_0} + \dfrac{\mu}{\Pr}$, $C_{P0} - C_{P\infty} = C_K + s(\tau_T - \tau_\rho)$. The condition of solvability of system (4) have a form

$$q_1^2 \left[(1 + \sigma_0 U\tau) q_2 + \frac{\sigma_0 U}{T_0} \frac{C_{P0} - C_{P\infty}}{C_{V\infty}} \right] + q_1 \frac{U^2 g_0}{T_0^2} \left[(1 + \sigma_0 U\tau) \frac{T_0'}{T_0} + \frac{E_0'}{C_{V\infty} T_0} \right] - \frac{U'}{T_0} \frac{2Ug_0}{T_0} \left[(1 + \sigma_0 U\tau) q_2 + \frac{\sigma_0 U}{T_0} \frac{C_{P0} - C_{P\infty}}{C_{V\infty}} \right] = 0.$$

In nonrelaxing ($\sigma_0 U\tau \to 0$) equilibrium ($S \to 0$) media equations (4) are coincided with equation founded in [2]. Since we have value of g_0:

$$g_0 = q_1^2 \frac{(1 + \sigma_0 U\tau) q_2 + \dfrac{\sigma_0 U}{T_0} \dfrac{C_{P0} - C_{P\infty}}{C_{V\infty}}}{\dfrac{U'}{T_0} \dfrac{2U}{T_0} \left[(1 + \sigma_0 U\tau) q_2 + \dfrac{\sigma_0 U}{T_0} \dfrac{C_{P0} - C_{P\infty}}{C_{V\infty}} \right] - \dfrac{q_1 U^2}{T_0^2} \left[(1 + \sigma_0 U\tau) \dfrac{T_0'}{T_0} + \dfrac{E_0'}{C_{V\infty} T_0} \right]}. \tag{5}$$

Equation (5) is connected the values y_+, g_0 and σ_0. One more equation for these parameters can be founded from next order of smallness. In result we have

$$q_1 \widehat{u}_1^{(1)} + \frac{U'}{T_0} \widehat{v}_1^{(0} = F_1(\xi), \quad \frac{2UG^2}{T_0} u_1^{(1)} + q_1 v_1^{(1)} - \frac{U^2 G^2}{T_0} T_1^{(1)} = F_2(\xi),$$

$$\left(\frac{T_0'}{T_0} + \frac{E_0'}{C_{V\infty} T_0} \right) v_1^{(1)} + q_2 T_1^{(1)} + \frac{\sigma_0 U}{C_{V\infty} T_0} E_1^{(1)} = \widetilde{F}_3(\xi),$$

$$\frac{E_0'}{T_0} \widehat{v}_1^{(1)} + \frac{\sigma U}{T_0} \widehat{E}_1^{(1)} - \frac{C_K \widehat{T}_1^{(1)} - \widehat{E}_1^{(1)}}{\tau T_0 C_{V\infty}} - q(\tau_T - \tau_\rho) \frac{T_1^{(1)}}{C_{V\infty} T_0^2} = F_4(\xi), \tag{6}$$

где $F_1 = -\eta q_1 u_1^{(0)} - \eta \left(\dfrac{U'}{T} \right)' v_1^{(0)}$, $F_2 = -\dfrac{2Ug}{T_0} u_1^{(0)} + g_1 \dfrac{U^{2\,\prime}}{T_0^2} T_1^{(0)} - 2g_0 \left(\dfrac{U}{T_0} \right)' \eta u_1^{(0)} - \eta q_1' v_1^{(0)} - g_0 \left(\dfrac{U^2}{T_0^2} \right)' \eta T_1^{(0)}$,

$$\widetilde{F}_3 = F_3 + \frac{F_4}{C_{V\infty}} = -\eta \left(\left(\frac{T_0'}{T_0} \right)' + \left(\frac{E_0'}{C_{V\infty} T_0} \right)' \right)' v_1^{(0)} - \eta q_2' T_1^{(0)} - \frac{\sigma_0 U}{C_{V\infty}} \left(\frac{U}{T_0} \right)' E_1^{(0)},$$

$$F_4 = -\eta \left(\frac{E_0'}{C_{V\infty} T_0} \right)' - \sigma_0 \eta \left(\frac{U}{T_0} \right)' E_1^{(0)} + \eta \left[\left(\frac{C_{P0} - C_{P\infty}}{\tau T_0} \right)' T_1^{(0)} - \left(\frac{1}{\tau T_0} \right)' E_1^{(0)} \right].$$

The conditions of solvability of system (6) are defined by Fredholm theorem. The system $\hat{A} \vec{\alpha}^{(1)} = \vec{F}$ have a non-trivial solution, if $\sum_{i=1}^{4} F_i \beta_i = 0$, where β_i are solutions of system $\hat{A}^T \vec{\beta}^{(1)} = 0$. The solutions of the last equations have a form $\beta_1 = -\dfrac{2Ug_0}{T_0} q_2$, $\beta_2 = q_1 q_2$,

$$\beta_3 = q_1 q_2 \frac{U^2 g_0}{T_0^2} \frac{1 + \sigma_0 U\tau}{(1 + \sigma_0 U\tau) q_2 + \dfrac{\sigma_0 U}{T_0} \dfrac{C_{P0} - C_{P\infty}}{C_{V\infty}}}, \quad \beta_4 = -q_1 q_2 \frac{U^2 g_0}{C_{V\infty} T_0^2} \frac{\sigma_0 U\tau}{(1 + \sigma_0 U\tau) q_2 + \dfrac{\sigma_0 U}{T_0} \dfrac{C_{P0} - C_{P\infty}}{C_{V\infty}}}.$$

Since

$$-\frac{2Ug_0}{T_0}q_2F_1+q_1q_2F_2+q_1q_2\frac{U^2g_0}{T_0^2}\frac{(1+\sigma_0U\tau)F_3}{(1+\sigma_0U\tau)q_2+\frac{\sigma_0U}{T_0}\frac{C_{P0}-C_{P\infty}}{C_{V\infty}}}-q_1q_2\frac{U^2g_0}{C_{V\infty}T_0^2}\frac{\sigma_0U\tau F_4}{(1+\sigma_0U\tau)q_2+\frac{\sigma_0U}{T_0}\frac{C_{P0}-C_{P\infty}}{C_{V\infty}}}=0. \quad (7)$$

The substitution of first order solution to equation (7) shows that in (7) presented only terms which proportional to η, or to g_1. In result we have $g_1 = 0$. The dependence of increment σ_0 from parameters y_+, g_0 in common case is unwieldy. The simplest form it have in limit $\frac{\sigma_0 U}{T_0} >> \mu$, which realized for great Reynolds and Dahmkeller numbers Re << 1 and Dh >> 1, fig.2.

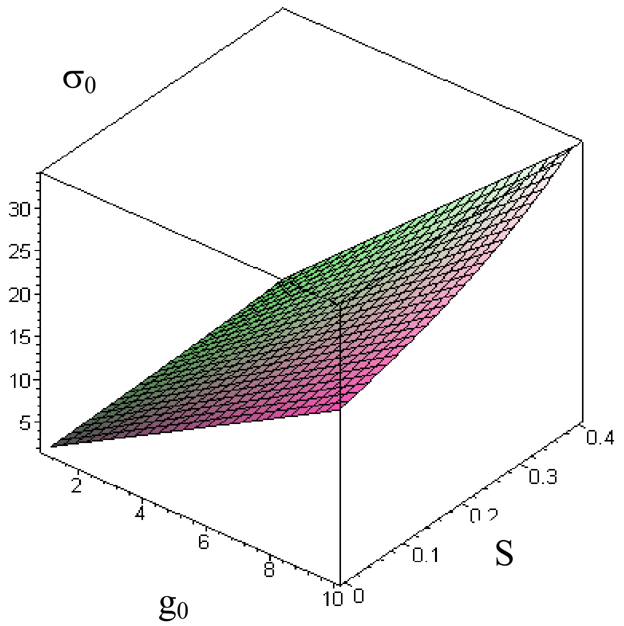

Fig.2 Dependence of increment σ_0 from Gertler number g_0 and degree of non-equilibrium S.

$$\sigma_0=\frac{g_0}{U}\left(2U'+\frac{\frac{T_0'}{T_0}+\frac{E_0'}{C_{V\infty}E_0}}{1+S\frac{\tau_T-\tau_\rho}{C_{V\infty}}}\right), \quad (8)$$

It well seen that increment of Gertler instability is sufficiently raised with growth of degree of non-equilibrium.

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