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AN ASYMPTOTIC APPROACH TO CALCULATION OF OSCILLATING COMPONENTS OF THE VELOCITY WAVE FIELD IN A TURBULENT FLOW UNDER HIGH REYNOLDS NUMBERS

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We develop a possible linearized asymptotic approach to predict turbulent components of the velocity field in a homogeneous flow at high Reynolds numbers. By an explicit view, the proposed approach looks like the Orr-Sommerfeld averaging method in theory of stability of laminar flows. However, the proposed method is different from that approach both by its physical essence and by the applied mathematical technique. In frames of a two-dimensional problem, there is applied a linearization of Navier-Stokes equations in dynamics of viscous incompressible fluid in a channel, when the turbulent oscillations of the velocity field are small when compared with the main averaged flow. The mean velocity is assumed to be known from the given fluid consumption. For simplicity, the unknown function of the through velocity distribution is approximated by a polyline. There are studied the regimes when oscillating and non-decaying in time solutions can arise. The considered linearized boundary value problem contains an unknown function of the averaged flow velocity. In this study we consider only regimes symmetric with respect the central line of the flow. The solution to the studied linearized boundary value problem is constructed asymptotically for high Reynolds numbers. After all this the so-constructed oscillating solution is substituted into averaged Reynolds equations that allows us to determine the width of the boundary layer, which is present as an unknown quantity in all equations as a result of the applied polyline approximation. Such an approach gives a clear instrument to the analytical definition of the sound field generated by a homogeneous turbulent flow.

Problem formulation. Let in a thin channel $|y| \le h$ of the width $2h$ be a flow of viscous incompressible fluid. Then the fluid motion is described by the Navier-Stokes equations and by the equation of continuity [1,2], which in terms of flow function have the form:

$$\frac{\partial}{\partial t}\Delta\psi + \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\Delta\psi - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\Delta\psi - \nu\Delta\Delta\psi = 0\,, \tag{1}$$

where ν is the kinematic viscosity, and the flow function ψ is introduced by the relations: $v_x = \dfrac{\partial\psi}{\partial y}$, $v_y = -\dfrac{\partial\psi}{\partial x}$, where $\bar{v} = \{v_x, v_y\}$ is the velocity vector of the fluid particles. The boundary conditions are the conditions of adhesion of the particles to the walls $v_x = 0$, $v_y = 0$, $y = \pm h$ that in terms of flow function is equivalent to the conditions:

$$\frac{\partial\psi}{\partial y} = 0\,,\ \frac{\partial\psi}{\partial x} = 0\,,\ y = \pm h\,. \tag{2}$$

Let us assume that the velocity field at any time moment t is a sum of the mean flow $U(y)$, directed along the length-wise coordinate x, and small perturbations, added to the mean stream:

$$v_x(x,y,t) = U(y) + v'_x(x,y,t)\,,\ v_y(x,y,t) = v'_y(x,y,t)\,, \tag{3}$$

where the main component of the velocity field $U(y)$ is assumed to be independent on time as well as on the through coordinate x, and the additional turbulent oscillations are assumed to be small when compared with the mean flow $U(y)$. This assumption is based on qualitative properties of the turbulent floes and is generally recognized [1–4]. In frames of such an approach the mean flow $U(y)$

is related to the flow function $\Psi(y)$: $U(y)=\frac{\partial\Psi}{\partial y}$, $\frac{\partial\Psi}{\partial x}=0\Rightarrow\Psi(y)=\int\limits_{-h}^{y}U(y)dy$, and then $\psi(x,y,t)=\Psi(y)+\psi'(x,y,t)$, где $v'_x(x,y,t)=\frac{\partial\psi'}{\partial y}, v'_y(x,y,t)=-\frac{\partial\psi'}{\partial x}$.

Now (1) takes the form:

$$\frac{\partial(\Delta\psi')}{\partial t}+\frac{\partial\Psi}{\partial y}\frac{\partial(\Delta\psi')}{\partial x}+\frac{\partial\psi'}{\partial y}\frac{\partial(\Delta\psi')}{\partial x}-\frac{\partial\psi'}{\partial x}\frac{\partial(\Delta\Psi)}{\partial y}-\frac{\partial\psi'}{\partial x}\frac{\partial(\Delta\psi')}{\partial y}-\nu\Delta\Delta\Psi-\nu\Delta\Delta\psi'=0 \quad (4)$$

Let us investigate this equation under the condition of the given flow consumption:

$$\int\limits_{-h}^{h}U(y)dy=D, \quad (5)$$

where the constant D is known being independent upon t and x.

Then the asymptotic analysis for high Reynolds numbers $\mathrm{Re}=2hU_0/\nu=D/\nu$ (where $U_0=D/(2h)$ is the average velocity over the profile) is equivalent to the assumption that the viscosity ν is small, i.e. Eq.(4) should be studied as $\nu\to 0$. In this case the next to last term in Eq. (4) can be neglected. The last term in that equation cannot be neglected since this reduces the true order (equal to four) of the differential equation. Besides, due to smallness of the turbulent components compared with the mean flow, we can neglect the both nonlinear terms.

Taking into account the above assumptions, Eq. (4) can be rewritten in the form:

$$\frac{\partial(\Delta\psi')}{\partial t}+U(y)\frac{\partial(\Delta\psi')}{\partial x}-U''(y)\frac{\partial\psi'}{\partial x}-\nu\Delta\Delta\psi'=0 \quad (6)$$

Since the mean flow will be constructed so that to satisfy the adhesion condition, the same conditions should be satisfied by the perturbed flow:

$$\frac{\partial\psi'}{\partial x}=\frac{\partial\psi'}{\partial y}=0,\ y=\pm h. \quad (7)$$

In the developed flow any moment of time can be formally chosen as an initial moment: $t=0$. It is assumed that such a turbulent flow is stable enough to small perturbations of the oscillating component of the velocity vector. If so, then this can be arbitrarily given as a small addition. Let us take the following initial condition:

$$v'_x(x,y,0)=\varepsilon\left(1-\frac{y^2}{h^2}\right)\cos(x/L),\ v'_y(x,y,0)=0, \quad (8)$$

where L is a certain characteristic size along the axis of the channel.

Symmetric solution. Let us apply the natural approach to the problem under consideration, when the unknown function of the lengthwise velocity distribution is approximated by a polyline. In the present work we restrict the consideration by the case of symmetric flow only (with respect to the channel axis $y=0$). Under such an approach the quantity $y_1=h-y_0$ designates the width of the boundary layer ($y_1<<h$), and the value of the velocity in the channel central zone $|y|\le y_0$ is asymptotically equal to the mean flow velocity U_0, due to an asymptotically narrow boundary layer and its asymptotically weak influence to the value of the integral in (5).

As soon as in the both zones $U''(y)=0$, hence Eq. (6) is equivalent

$$\frac{\partial(\Delta\psi')}{\partial t}+U(y)\frac{\partial(\Delta\psi')}{\partial x}-\nu\Delta\Delta\psi'=0. \quad (9)$$

Let us apply to Eq.(9) the Laplace transform with respect to t:

$$\psi'(x,y,p)=\int\limits_0^{\infty}\psi'(x,y,t)e^{-pt}dt,\quad \psi'(x,y,t)=\frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty}\psi'(x,y,p)e^{pt}dp. \quad (10)$$

Then Eq. (9) becomes

$$p\Delta\psi'+U(y)\frac{\partial(\Delta\psi')}{\partial x}-\nu\Delta\Delta\psi'=\Delta\psi'\big|_{t=0}=-\frac{2\varepsilon}{h^2}y\cos(x/L)=-\frac{2\varepsilon}{h^2}y\left(e^{ix/L}+e^{-ix/L}\right) \qquad (11)$$

It can now be easily seen that due to linearity of Eq. (11):

$$\psi'(x,y,p)=\psi'_L(y,p)e^{ix/L}+\psi'_{-L}(y,p)e^{-ix/L}. \qquad (12)$$

For some time, let us omit parameter p in arguments of all functions, for the sake of brevity, considering it as a parameter. For the same reason, let perform all mathematical transformations only for the case $e^{ix/L}$. The case $e^{-ix/L}$ can be easily obtained from all further formulas replacing L by $-L$. For all that we omit the subscript $\pm L$. Due to the symmetry, it is sufficient to study the zone $y \ge 0$ only. In the central part of the channel the solution has the form:

$$\psi'_1(y)=L\int_0^y sh\frac{y-\eta}{L}\left[C_5\frac{sh(\gamma_0\eta)}{\sqrt{\gamma_0}}-\frac{\varepsilon}{h^2\nu\gamma_0^2}\right]d\eta+C_6 L sh\frac{y}{L},\ 0\le y\le y_0, \qquad (13)$$

where $\gamma_0=\sqrt{\frac{p}{\nu}+\frac{iU_0}{L\nu}+\frac{1}{L^2}}$. Obviously, from two possible branches of the root square we choose the arithmetic branch, for which $\operatorname{Re}(\gamma_0)\ge 0$.

On the interval $y_0 \le y \le h$ it is convenient to introduce a local variable $\tilde{y}=h-y$, $0\le\tilde{y}\le y_1$. Then, with the use of the WKB-method [5], we can construct the following solution in the boundary layer:

$$\psi'_2(y)=L\int_0^{\tilde{y}} sh\frac{\tilde{y}-\eta}{L}\left[C_1\frac{e^{\xi(\eta)}}{\sqrt{\gamma(\eta)}}+C_2\frac{e^{-\xi(\eta)}}{\sqrt{\gamma(\eta)}}-\frac{\varepsilon}{h^2\nu}\frac{h-\eta}{\gamma^2(\eta)}\right]d\eta+C_3 sh\frac{\tilde{y}}{L}+C_4 ch\frac{\tilde{y}}{L},$$

$$\xi(\tilde{y})=\int_0^{\tilde{y}}\gamma(\eta)d\eta,\ \gamma(\tilde{y})=\sqrt{\frac{p}{\nu}+\frac{iU_0\tilde{y}}{L\nu y_1}+\frac{1}{L^2}},\ 0\le\tilde{y}\le y_1, \qquad (14)$$

It follows from the boundary conditions (8) that $C_3=C_4=0$. By using an asymptotic estimate of some integrals by the Laplace method [6], from the boundary conditions we can obtain the following 4×4 linear algebraic system for the unknown quantities (C_1, C_2, C_5, C_6):

$$\left(\begin{array}{cccc|c}
\dfrac{e^{\xi_1}}{\gamma_0^{5/2}L} & \dfrac{sh(y_1/L)}{\left(\dfrac{p}{\nu}+\dfrac{1}{L^2}\right)^{3/4}} & -a_s & -sh\dfrac{y_0}{L} & f_1 \\
\dfrac{e^{\xi_1}}{\gamma_0^{3/2}} & \dfrac{ch(y_1/L)}{\left(\dfrac{p}{\nu}+\dfrac{1}{L^2}\right)^{3/4}} & b_s & ch\dfrac{y_0}{L} & f_2 \\
e^{\xi_1} & e^{-\xi_1} & -sh(\gamma_0 y_0) & 0 & 0 \\
e^{\xi_1}\left(1-\dfrac{iU_0}{4\nu y_1 L\gamma_0^3}\right) & -e^{-\xi_1}\left(1+\dfrac{iU_0}{4\nu y_1 L\gamma_0^3}\right) & ch(\gamma_0 y_0) & 0 & f_3
\end{array}\right), \qquad (15)$$

where the following notations are introduced for integrals, which can be calculated explicitly:

$$a_s=\frac{1}{\sqrt{\gamma_0}}\int_0^{y_0} sh\frac{y_0-\eta}{L}sh(\gamma_0\eta)d\eta,\ b_s=\frac{1}{\sqrt{\gamma_0}}\int_0^{y_0} ch\frac{y_0-\eta}{L}sh(\gamma_0\eta)d\eta,\ f_3=-\frac{i\varepsilon U_0 y_0}{h^2\nu^2\gamma_0^{9/2}Ly_1},$$

$$f_1=\frac{\varepsilon}{h^2\nu}\int_0^{y_1} sh\frac{y_1-\eta}{L}\frac{h-\eta}{\gamma^2(\eta)}d\eta-\frac{\varepsilon}{h^2\nu\gamma_0^2}\int_0^{y_0} sh\frac{y_0-\eta}{L}\eta d\eta, \qquad (16)$$

$$f_2 = \frac{\varepsilon}{h^2\nu}\int_0^{y_1} ch\frac{y_1-\eta}{L}\frac{h-\eta}{\gamma^2(\eta)}d\eta + \frac{\varepsilon}{h^2\nu\gamma_0^2}\int_0^{y_0} ch\frac{y_0-\eta}{L}\eta d\eta\,.$$

The principal determinant of this system is

$$\Delta = \frac{1}{\gamma_0^{3/2}}\left[sh\frac{y_0}{L}+\frac{1}{\gamma_0 L}ch\frac{y_0}{L}\right]\left[ch(\gamma_0 y_0)-\left(1+\frac{iU_0}{4\nu y_1 L\gamma_0^3}\right)sh(\gamma_0 y_0)\right]- \qquad (17)$$

$$-e^{\xi_1}\frac{sh(h/L)}{\left(\frac{p}{\nu}+\frac{1}{L^2}\right)^{3/4}}\left[ch(\gamma_0 y_0)+\left(1-\frac{iU_0}{4\nu y_1 L\gamma_0^3}\right)sh(\gamma_0 y_0)\right]-\frac{1}{\sqrt{\gamma_0}}\left\{\frac{sh\left[y_0(\gamma_0+1/L)\right]}{\gamma_0+1/L}-\frac{sh\left[y_0(\gamma_0-1/L)\right]}{\gamma_0-1/L}\right\}$$

Hence, the solution to this system (17) can be constructed by using the Kramer's rule: $C_1=\Delta_1/\Delta$, $C_2=\Delta_2/\Delta$, $C_5=\Delta_3/\Delta$, $C_6=\Delta_4/\Delta$, where Δ_i, $i=1,\dots,4$ are particular determinants, which are obtained from the principal one by the change of the i-th column by the vector from the right-hand side.

The substitution of these relations to (13) and (14) allows us to express the solution both in the mean and in the boundary-layer zones in its explicit dependence upon Laplace parameter p. The inversion of the Laplace transform by the second formula (10) shows that the deformation of the integration contour to the left half-plane $\mathrm{Re}(p)\le 0$ results in a decaying with time solution, if the principal determinant $\Delta(p)$ has no zeroes on the imaginary axis [7]. The equation $\Delta(p)=0$ generates the sets ω_n, L_n, where ω_n is real-valued and $\Delta(i\omega_n)=0$. Then we can extract from the system (15) C_5, C_6 $:C_5=\frac{\Delta_3(p)}{\Delta(p)}=\frac{(\gamma_0^2-1/L^2)\sqrt{\gamma_0}}{L}\frac{C_5^*(p)}{\Delta(p)}$, $C_6=\frac{\Delta_4(p)}{\Delta(p)}=\frac{1}{L}\frac{C_6^*(p)}{\Delta(p)}$, that leads to the asymptotic representation of the function ψ'_1 for large time:

$$\psi'_1(x,y,t)\sim\left.\frac{C_6^*(i\omega_n)}{\Delta'(i\omega_n)}\right|_{L=L_n} e^{i(x/L+\omega_n t)}sh\frac{y}{L}+\left.\frac{C_5^*(i\omega_n)}{\Delta'(i\omega_n)}\right|_{L=L_n} e^{i(x/L+\omega_n t)}\left[\gamma_0 sh(\gamma_0 y)-\frac{1}{L}sh\frac{y}{L}\right]-$$

$$-\left.\frac{C_6^*(-i\omega_n)}{\Delta'(-i\omega_n)}\right|_{L=-L_n} e^{-i(x/L+\omega_n t)}sh\frac{y}{L}+\left.\frac{C_5^*(-i\omega_n)}{\Delta'(-i\omega_n)}\right|_{L=-L_n} e^{-i(x/L+\omega_n t)}\left[\gamma_0 sh(\gamma_0 y)-\frac{1}{L}sh\frac{y}{L}\right]. \qquad (18)$$

By analogy, we can write out a respective representation for ψ'_2 too.

Now one can perform the averaging of the nonlinear equations (3) in the central zone of the channel:

$$\overline{\frac{\partial\psi'_1}{\partial y}\frac{\partial(\Delta\psi'_1)}{\partial x}}=\overline{\frac{\partial\psi'_1}{\partial x}\frac{\partial(\Delta\psi'_1)}{\partial y}}\,. \qquad (19)$$

This equation (19) allows us to obtain the value of the width y_1 of the boundary layer.

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REFERENCES

1. Loytsiansky L.G. Mechanics of Fluid and Gas. – Nauka: Moscow, 1970. – 904 p. (in Russian).
2. Schlichting H. Grenzschicht-Theorie. Verlag G.Braun: Karlsruhe /5 ed./, 1964.
3. Kochin N.E., Kibel' I.A., Rose N.V. Theoretical Hydromechanics, v.2 – Fizmatgiz: Moscow, 1963. – 728 p.
4. Hydrodynamic Instabilities and the Transition to Turbulence /ed. Swinney H.L., Gollub J.P./, Springer-Verlag: Heidelberg, 1981.
5. Fedoryuk M.V. Asymptotic Methods for Linear Ordinary Differential Equations. – Nauka: Moscow, 1983. – 352 p. (in Russian).
6. Fedoryuk M.V. Asymptotics: Integrals and Series. – Nauka: Moscow, 1987. – 544 p. (in Russian).
7. Ditkin V.A., Prudnikov A.P. Integral Transformations and Operational Calculus. – Nauka: M, 1974. – 542 p.